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SUMMARY

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The problem of the transport of thermal radiation through an absorbing medium between parallel walls held at fixed different temperatures is considered. The method of solution is based on a procedure introduced by Yvon in the analysis of monoenergetic neutron transport and employs double spherical harmonics expansions of the specific intensity. Formulas for the energy transport between the walls and the temperature distribution in the medium are obtained for a simple approximation. Numerical results are obtained for a more complicated approximation. The results are compared to previously published values.

AUTHOR

INTRODUCTION

There are many similarities between the mathematical description of the transfer of radiation in a gas and the transfer of neutrons in a medium. These have been discussed in reference 1, page 37. If certain simplifying assumptions are made for both the neutron and the radiation-transfer equation, then the two transfer equations have the same form.

For this reason the methods developed either for radiation or neutron analysis can be applied to the other field. One procedure widely used in neutron transport analysis is the so-called double spherical harmonics method introduced by Yvon (ref. 2).

Yvon's method is a variation of the well-known, full-range spherical harmonics approximation (see, e.g., ref. 1 or ref. 3). In this full-range solution the angular distribution of the intensity is expanded in a series of Legendre polynomials. The full-range results are known to be relatively accurate in the interior of a large homogeneous medium where the specific intensity is generally a slowly varying function of angle. At boundaries between different media a discontinuity in the intensity occurs, and it is at such points that large errors may arise. This is due to the difficulty in expressing a discontinuous function in terms of a series of continuous functions.

To improve the convergence of the solution at boundaries, Yvon followed the procedure of considering separately the components of the radiation traveling in the positive and negative directions. Each component is expanded in an independent series of Legendre polynomials. This allows the solution to have discontinuities at the boundaries as demanded by the boundary conditions. Thus the double spherical harmonics solution has been found to be more accurate than the full-range result near discontinuities in a medium, or when applied to optically thin media. Yvon's method, as presented in this paper, is applicable only to one-dimensional problems.

Both Shiff and Ziering (ref. 4) and Bengston (ref. 5) have developed Yvon's method and applied it to problems in neutron transport. In reference 4 the accuracy has been investigated in some detail. It is found, for example, that for Milne's problem the double spherical harmonics approximation gives better results than the full-range approximation of comparable complexity. The extension of the double expansion procedure to two-dimensional problems has also been considered by Shiff and Ziering (ref. 6).

It is the purpose of this paper to apply Yvon's method to the problem of the plane absorbing layer of grey gas between absorbing and emitting walls at fixed different temperatures. The walls are assumed to radiate isotropically and to have an absorptivity of unity (i.e., black walls). First, reduction of the integro-differential equation describing the physical situation to a set of ordinary differential equations is illustrated. The boundary conditions at the walls must be introduced to complete the solution of these differential equations. Finally, calculations giving such quantities as the energy flux across the gas slab and the temperature distribution through that slab are presented and compared with previously published values.

SYMBOLS

A^+,B^+ coefficients in equation (A18)

A^-,B^- coefficients in equation (A19)

a_i coefficient in equation (30)

d_n coefficient in equation (15)

f_n,h_n coefficients in equation (16)

g_n $\dfrac{d_n}{a_i}$

$I(\xi,\mu)$ specific intensity per unit area, time, solid angle

$I_n(\xi)$ coefficients in the series expansion of $I(\xi,\mu)$ (see eqs. (7) and (8))

k coefficient in exponent (see discussion preceding eq. (13))

L geometric thickness of plane layer (see fig. 1)

P_n Legendre polynomial of the first kind of order n

q rate of energy transport per unit area

r reflectivity

T temperature, absolute

W_n nonsingular part of the Legendre function of the second kind of order n

x geometric depth in absorbing layer

$\beta(\xi)$ emission function (see eq. (35))

ϵ emissivity

θ angle between ray and normal to surface (see fig. 1)

κ local absorption coefficient, per unit mass

λ $\frac{1}{k}$

μ $\cos \theta$

ξ optical depth in absorbing layer ($d\xi = \rho\kappa\, dx$)

ρ local density of absorbing medium

σ Stefan-Boltzmann constant

$\varphi(\xi)$ dimensionless form of emission function (see eq. (36))

Subscripts

L evaluated at $\xi = \xi_L = \int_0^L \rho\kappa\, dx$

N maximum value of n in expansion

n order of Legendre polynomial or function

o evaluated at $\xi = 0$

Superscripts

+ right-going (ξ-increasing) quantity

- left-going (ξ-decreasing) quantity

GENERAL SOLUTION

Integro-Differential Equation

The one-dimensional, time-independent, radiation-transfer equation for a grey gas in local thermodynamic equilibrium is normally written (see ref. 1) as

$$\mu \frac{dI(x,\mu)}{\rho\kappa \ dx} + I(x,\mu) = \frac{\sigma T(x)^4}{\pi} \tag{1}$$

The specific intensity, $I(x,\mu)$, is the energy flux per unit area, time, and solid angle, at point x in direction μ. (For a more detailed discussion of specific intensity consult either reference 1 or reference 7.) The symbols T, σ, and x are, respectively, the local temperature, Stefan-Boltzmann constant, and the geometric length. Mass absorption coefficient and density are designated by κ and ρ, and θ ($= \cos^{-1} \mu$) is the angle between the direction of radiation and the x axis (refer to fig. 1 for the coordinate system and the geometry).

The optical depth ξ of the medium is defined in terms of ρ, κ, and x by the following equation.

$$d\xi = \rho\kappa \ dx$$

Equation (1) can then be written in terms of the optical depth as

$$\mu \frac{dI(\xi,\mu)}{d\xi} + I(\xi,\mu) = \frac{\sigma T(\xi)^4}{\pi} \tag{2}$$

The above formulation of the grey gas radiation-transfer equation is based on the assumption of local thermodynamic equilibrium. Another formulation, corresponding to perfect isotropic scattering, is shown in reference 1, page 32, to be mathematically equivalent. This equation is

$$\mu \frac{dI(\xi,\mu)}{d\xi} + I(\xi,\mu) = \frac{1}{2}\int_{-1}^{1} I(\xi,\mu)d\mu \tag{3}$$

There are two unknowns in equation (2), $T(\xi)$ and $I(\xi,\mu)$. In equation (3) the only unknown is $I(\xi,\mu)$, and the equation is now an integro-differential equation.

Equation (3) has the same form as the energy independent, one-dimensional neutron-transport equation for a nonabsorbing medium. Yvon's method will be applied to equation (3). In the problem treated here, the walls are held at different fixed temperatures. The mathematical relation between the specific intensity $I(\xi,\mu)$ and the temperature $T(\xi)$ is obtained by equating the right sides of equations (2) and (3).

Application of Yvon's Method

In the application of Yvon's method the components of the radiation traveling in the positive and negative directions are considered separately. Each of these components is then expanded in a series of Legendre polynomials[1] and substituted into the integro-differential equation. A set of ordinary differential equations is obtained. The following notation will be used to differentiate between the positive and negative components of the radiation.

$$I(\xi,\mu) = \left.\begin{array}{ll} I^+(\xi,\mu) & 0 \leq \mu \leq 1 \\ I^-(\xi,\mu) & -1 \leq \mu \leq 0 \end{array}\right\} \tag{4}$$

It will be necessary to use one of the recurrence formulas for Legendre polynomials as well as the orthogonality conditions for the polynomials which apply to the ranges $0 \leq \mu \leq 1$ and $-1 \leq \mu \leq 0$. These are, respectively,

$$(2n + 1)P_1(y)P_n(y) = (n + 1)P_{n+1}(y) + nP_{n-1}(y) \tag{5}$$

$$\left.\begin{array}{l} \displaystyle\int_0^1 P_n(2\mu - 1)P_m(2\mu - 1)d\mu = \frac{1}{2n + 1}\delta_{nm} \\ \displaystyle\int_{-1}^0 P_n(2\mu + 1)P_m(2\mu + 1)d\mu = \frac{1}{2n + 1}\delta_{nm} \end{array}\right\} \tag{6}$$

where $P_n(y)$ is the Legendre polynomial of order n with argument y and where δ_{nm} is the Kronecker delta defined as follows.

$$\begin{aligned} \delta_{nm} &= 1 , \quad n = m \\ &= 0 , \quad n \neq m \end{aligned}$$

---

[1]Yvon's method is also called the double spherical harmonics method although the Legendre polynomials are used in this paper instead of the spherical harmonics. For cases where θ is the only angle variable, the spherical harmonics and the Legendre polynomials are proportional.

Equations (6) are obtained directly from the normal orthogonality formulas for the Legendre polynomials by a simple transformation of variables.

The procedure by which a set of ordinary first-order differential equations is obtained from the integro-differential equation has been included in other papers (refs. 2, 4, and 5) and, therefore, will be only briefly outlined below. The general method is first to expand $I^+(\xi,\mu)$ and $I^-(\xi,\mu)$ in infinite series of the polynomials $P_n(2\mu - 1)$ and $P_n(2\mu + 1)$. These series are then substituted into equation (3) and after some rearrangement it is possible to equate the coefficients of like terms in the series which appear on both sides of the resulting equation. The coefficients contain first derivatives of $I_n^+(\xi)$ and $I_n^-(\xi)$. Therefore, by equating each pair of terms of the series, we obtain one ordinary differential equation.

The expansions of the specific intensities are

$$I^+(\xi,\mu) = \sum_n (2n + 1)P_n(2\mu - 1)I_n^+(\xi) \tag{7}$$

$$I^-(\xi,\mu) = \sum_n (2n + 1)P_n(2\mu + 1)I_n^-(\xi) \tag{8}$$

The (2n + 1) factor is included in equations (7) and (8) for convenience in later manipulations.

Equations (7) and (8) are substituted into equation (3). Employing the orthogonality relations (eqs. (6)) and the recurrence formula (eq. (5)), the following equations are obtained.

$$\frac{1}{2}\sum_n P_n(2\mu - 1)\left[n\,\frac{dI_{n-1}^+}{d\xi} + (n + 1)\,\frac{dI_{n+1}^+}{d\xi} + (2n + 1)\,\frac{dI_n^+}{d\xi}\right] + \sum_n (2n + 1)P_n(2\mu - 1)I_n^+ - \frac{1}{2}\left(I_o^- + I_o^+\right) = 0 \tag{9}$$

$$\frac{1}{2}\sum_n P_n(2\mu + 1)\left[n\,\frac{dI_{n-1}^-}{d\xi} + (n + 1)\,\frac{dI_{n+1}^-}{d\xi} - (2n + 1)\,\frac{dI_n^-}{d\xi}\right] + \sum_n (2n + 1)P_n(2\mu + 1)I_n^- - \frac{1}{2}\left(I_o^+ + I_o^-\right) = 0 \tag{10}$$

It is to be understood that I_n^+ and I_n^- are always functions of the optical depth ξ. The coefficients I_n^+ and I_n^- are taken to be zero when $n < 0$.

The following two differential equations are obtained by equating to zero the coefficients of $P_n(2\mu - 1)$ and $P_n(2\mu + 1)$ in equations (9) and (10).

$$\frac{d}{d\xi}\left[nI_{n-1}^{+} + (2n + 1)I_n^{+} + (n + 1)I_{n+1}^{+}\right] + 2(2n + 1)I_n^{+} - \left(I_o^{+} + I_o^{-}\right)\delta_{no} = 0$$

$$\text{for} \quad 0 \leq \mu \leq 1 \qquad (11)$$

$$\frac{d}{d\xi}\left[nI_{n-1}^{-} - (2n + 1)I_n^{-} + (n + 1)I_{n+1}^{-}\right] + 2(2n + 1)I_n^{-} - \left(I_o^{+} + I_o^{-}\right)\delta_{no} = 0$$

$$\text{for} \quad -1 \leq \mu \leq 0 \qquad (12)$$

δ_{lo} is the Kronecker delta as defined previously.

In the double P_N approximation it is assumed that I_n^{+} and I_n^{-} are zero for $n \geq N + 1$. This is the same as assuming that the expansions for $I^{+}(\xi,\mu)$ and $I^{-}(\xi,\mu)$ (eqs. (7) and (8)) contain only $(N + 1)$ terms. For the general case this is, of course, an approximation.

The problem now consists of solving the set of $2(N + 1)$ coupled ordinary differential equation, represented by equations (11) and (12), subject to specified boundary conditions. First, however, solutions of the differential equations are obtained which contain undetermined constants. The boundary conditions are used later to determine these constants.

Characteristic Determinant

Some information about the solutions of equations (11) and (12) can be obtained from the characteristic determinant for these equations without actually solving for the roots of that determinant. If the usual exponential solution e^{kx} is substituted into equations (11) and (12), the following characteristic determinant is obtained.

$$\begin{vmatrix} 1 + k & -1 & k & 0 & 0 & 0 & \cdots \\ -1 & 1 - k & 0 & k & 0 & 0 & \\ k & 0 & 6 + 3k & 0 & 2k & 0 & \\ 0 & k & 0 & 6 - 3k & 0 & 2k & \\ 0 & 0 & 2k & 0 & 10 + 5k & 0 & \\ 0 & 0 & 0 & 2k & 0 & 10 - 5k & \\ \vdots & & & & & & \ddots \end{vmatrix} = 0 \qquad (13)$$

The equations were arranged in order of increasing n with the I_n^+ and I_n^- equations alternating, the I_n^+ before the I_n^-.

By elementary operations equation (13) can be reduced to

$$k^2 \begin{vmatrix} 7+2k & 1 & 2k & 0 & 0 & \cdots \\ 1 & 7-2k & 0 & 2k & 0 & \\ 2k & 0 & 10+5k & 0 & 3k & \\ 0 & 2k & 0 & 10-5k & 0 & \\ 0 & 0 & 3k & 0 & 14+7k & \\ \vdots & & & & & \ddots \end{vmatrix} = 0 \qquad (14)$$

Equation (14) shows there are two zero roots of the characteristic equation (13). Zero is not a root of the determinant in equation (14) as can be demonstrated by substitution of $k = 0$ into the determinant. Thus, there are only two zero roots of equation (13) and these have been factored in equation (14).

Each row of the determinant in equation (14) can now be divided by k. Let $\lambda = 1/k$. Each element of the resulting determinant is of the form $A_{ij} + \lambda C_{ij}$ where C_{ij} are the elements of a positive definite matrix and A_{ij} are the elements of a real symmetric matrix. Then by theorem 44 of reference 8 all the roots λ_i are real.

Solution of the Differential Equations

It is now known that each solution of the $2(N + 1)$ differential equations (i.e., $I_n^{\pm}$) will consist of a constant and a linear term, and $2N$ exponential terms with real exponents. The exponential terms are of the form

$$d_n^{\pm}\, e^{-\xi/\lambda_i} \qquad (15)$$

and the constant and linear solutions are of the form

$$f_n^{\pm} + h_n^{\pm}\xi \qquad (16)$$

where the constants $d_n^{\pm}$, $f_n^{\pm}$, and $h_n^{\pm}$ are functions only of the particular λ_i. The solution for each of the $2(N + 1)$ values of $I_n^{\pm}$ will be of the form

$$I_n^{\pm} = f_n^{\pm} + h_n^{\pm}\xi + \sum_{i=1}^{2N} d_n^{\pm} e^{-\xi/\lambda_i} \tag{17}$$

In the double P_N approximation there are $[2(N + 1)]^2$ constants (i.e., $d_n^{\pm}$, $f_n^{\pm}$, and $h_n^{\pm}$) and 2N values of the exponents λ_i to be determined. The first step in the solution is to substitute equations (15) and (16) into the differential equations. Because of the coupling in the equations, enough relations among the constants $d_n^{\pm}$, $f_n^{\pm}$, and $h_n^{\pm}$ are obtained such that λ_i and all but $2(N + 1)$ of the constants can be determined. The details of this solution, which is fairly complex algebraically, are contained in appendix A. The boundary conditions are used to determine the remaining $2(N + 1)$ constants. Application of the boundary conditions is discussed in the next section.

The procedure used in appendix A to obtain λ_i and the constants $d_n^{\pm}$, $f_n^{\pm}$, and $h_n^{\pm}$ is somewhat different from the methods employed in references 2, 4, and 5. The present procedure, which is similar to that used in reference 3 for the solution of the differential equations obtained in the full-range spherical harmonics problem, appears to offer some reduction in the complexity of the solution for higher approximations. This reduction in complexity results from the use of the recurrence formulas for the Legendre polynomials to generate the values of $d_n^{\pm}$ from the preceding $d_{n-1}^{\pm}$ and $d_{n-2}^{\pm}$. Also, the use of the recurrence formulas leads directly to a polynomial (eq. A27)), the roots of which are the 2N values of the exponents λ_i.

APPLICATION OF BOUNDARY CONDITIONS

The problem of the plane absorbing layer of gas between absorbing and emitting walls is to be considered. The walls are held at fixed different temperatures and are assumed to be black. This assumption involves no loss in generality since as shown, for example, in reference 9, the solutions for the black wall and the grey wall cases are related. Figure 1 shows the geometry and the coordinate system.

The intensity of the radiation from an isotropically radiating black wall at temperature T is $\sigma T^4/\pi$. The walls are at $\xi = 0$ and $\xi = \xi_L$. Then

$$\frac{\sigma T_o^4}{\pi} = I^+(0,\mu) = \sum_n (2n + 1)P_n(2\mu - 1)I_n^+(0) \tag{18}$$

$$\frac{\sigma T_L^4}{\pi} = I^-(\xi_L,\mu) = \sum_n (2n + 1)P_n(2\mu + 1)I_n^-(\xi_L) \tag{19}$$

The left sides of equations (18) and (19) are not functions of μ. Therefore, the terms $I_n^+(0)$ and $I_n^-(\xi_L)$ must be zero for $n \geq 1$ since P_n are functions of μ for $n \geq 1$. The boundary conditions are then

$$\frac{\sigma T_o^4}{\pi} = I_o^+(0) \tag{20}$$

$$0 = I_n^+(0) , \quad 1 \leq n \leq N \tag{21}$$

$$\frac{\sigma T_L^4}{\pi} = I_o^-(\xi_L) \tag{22}$$

$$0 = I_n^-(\xi_L) , \quad 1 \leq n \leq N \tag{23}$$

so that $2(N + 1)$ equations in $I_n^+(0)$ and $I_n^-(\xi_L)$ are obtained. Thus the boundary conditions provide the correct number of relations to evaluate the $2(N + 1)$ as yet undetermined constants.

For isotropically radiating walls, $I^+(0,u)$ and $I^-(\xi_L,\mu)$ are constants with respect to the angle variable μ. It is mathematically possible to obtain solutions to the problem when $I^+(0,u)$ and $I^-(\xi_L,u)$ are functions of μ. In this case the values of $I_n^+(0)$ and $I_n^-(\xi_L)$ are determined from equations (18) and (19) and the orthogonality relations for the Legendre polynomials. The physical meaning of such boundary conditions and their relation to wall temperature has not been investigated and will not be considered again in this paper.

PLANE ABSORBING LAYER-ENERGY FLUX AND TEMPERATURE

Energy Flux

The positive flux $q^+(\xi)$ and the negative flux $q^-(\xi)$ are defined as the total rates of energy transport per unit area in the positive and negative directions, respectively. The expressions for these fluxes in terms of the specific intensities are given in reference 10. They are

$$q^+(\xi) = 2\pi \int_0^1 I^+(\xi,\mu)\mu \, d\mu \tag{24}$$

$$q^-(\xi) = 2\pi \int_0^{-1} I^-(\xi,\mu)\mu \, d\mu \tag{25}$$

and

$$q(\xi) = q^+(\xi) - q^-(\xi) \tag{26}$$

where $q(\xi)$ is the local net value of the energy flux. A net energy flow in the positive x direction corresponds to $q(\xi) > 0$.

It is possible to write $q^+(\xi)$ as a function of $I_n^+(\xi)$. Substitute equation (7) into equation (24) and let $\mu = (1/2)[P_1(2u - 1) + P_0(2u - 1)]$. The orthogonality relation (eq. (5)) gives

$$q^+(\xi) = \pi(I_0^+ + I^+) \tag{27}$$

and similarly

$$q^-(\xi) = \pi(I_0^- - I_1^-) \tag{28}$$

and from equation (26)

$$q(\xi) = \pi(I_0^+ + I_1^+ - I_0^- + I_1^-) \tag{29}$$

Equations (A28) and (A7) through (A10) are substituted into equation (29) to get the flux in terms of a_i, $g_n^{\pm}$, and h_0^+ (as defined in appendix A, $d_n^{\pm}(\lambda_i) = a_i g_n^{\pm}(\lambda_i)$).

$$\frac{q(\xi)}{\pi} = -\frac{4}{3}h_0^+ + \sum_{i=1}^{2N} a_i\left[g_0^+(\lambda_i) + g_1^+(\lambda_i) - g_0^-(\lambda_i) + g_1^-(\lambda_i)\right]e^{-\xi/\lambda_i} \tag{30}$$

Note that the linear term in equation (30) has disappeared. If equations (A20), (A25), and (A26) are substituted into equation (30), the term inside the brackets reduces to zero. Therefore

$$q = -\frac{4}{3}\pi h_0^+ \tag{31}$$

Since h_0^+ is not a function of ξ, the value of the energy flux is constant with respect to position. This is a condition that must be true for the exact solution and is, of course, desirable in any approximate solution such as the one being considered here. Note that h_0^+ is a function of both the order of the approximation and the boundary conditions.

Temperature

At a point between the walls, the temperature and the specific intensity are related by the following equation:

$$\frac{1}{2}\int_{-1}^{1} I(\xi,\mu)d\mu = \frac{\sigma T(\xi)^4}{\pi} \tag{32}$$

Equation (32) follows from equations (1) and (3). The left side of (32) can be evaluated:

$$\frac{1}{2}\int_{-1}^{1} I(\xi,\mu)d\mu = \frac{1}{2}\int_{-1}^{0} I^{-}(\xi,\mu)d\mu + \frac{1}{2}\int_{0}^{1} I^{+}(\xi,\mu)d\mu \tag{33}$$

From equations (5) through (8) this is

$$\frac{1}{2}\int_{-1}^{1} I(\xi,\mu)d\mu = \frac{1}{2}\left(I_0^{+} + I_0^{-}\right) \tag{34}$$

It is customary to define a function, $\beta(\xi)$, the emission function, as

$$\beta(\xi) = \sigma T(\xi)^4 = \frac{\pi}{2}\left(I_0^{+} + I_0^{-}\right) \tag{35}$$

Heaslet and Fuller (ref. 9) have also introduced the function $\varphi(\xi)$ defined below.

$$\varphi(\xi) = \frac{\beta(\xi)}{q^{-}(\xi_L) - q^{+}(0)} - \frac{q^{+}(0) + q^{-}(\xi_L)}{2[q^{-}(\xi_L) - q^{+}(0)]} \tag{36}$$

The function $\varphi(\xi)$ can be considered as a normalized, dimensionless $\beta(\xi)$. The usefulness of the function $\varphi(\xi)$ is as follows. The problem is first solved and $\varphi(\xi)$ calculated for black walls. From this value of $\varphi(\xi)$ it is then possible, using the method of reference 9, to calculate $\beta(\xi)$ for the case with grey walls. This is equivalent to having the following boundary conditions:

$$I_0^{+}(0) = \frac{1}{\pi}\left[\epsilon_0 \sigma T_0^{\,4} + r_0 q^{-}(0)\right] \tag{37}$$

$$I_1^{+}(0) = 0 \tag{38}$$

$$I_0^{-}(\xi_L) = \frac{1}{\pi}\left[\epsilon_L \sigma T_L^{\,4} + r_L q^{+}(\xi_L)\right] \tag{39}$$

$$I_1^{-}(\xi_L) = 0 \tag{40}$$

where ϵ_0 and ϵ_L are the emissivities and r_0 and r_L are the reflectivities of the left and right walls, respectively.

Double P_1 Approximation

This example, which involves only four differential equations and four unknowns, is interesting since it is simple enough to be solved in terms of an arbitrary optical thickness, ξ_L, of the absorbing layer. The differential equations are

$$\frac{d}{d\xi}\left(I_1^+ + I_o^-\right) + \left(I_o^+ - I_o^-\right) = 0 \tag{41}$$

$$\frac{d}{d\xi}\left(I_o^+ + 3I_1^+\right) + 6I_1^+ = 0 \tag{42}$$

$$\frac{d}{d\xi}\left(I_1^- - I_o^-\right) + \left(I_o^- - I_o^+\right) = 0 \tag{43}$$

$$\frac{d}{d\xi}\left(I_o^- - 3I_1^-\right) + 6I_1^- = 0 \tag{44}$$

and the boundary conditions are equations (20) through (23) with $N = 1$. The mathematics leading to the solution is lengthy but straightforward. The results obtained are

$$\frac{\beta(\xi)}{q^-(\xi_L) - q^+(0)} = \frac{\frac{\pi}{2}\left(I_o^+ + I_o^-\right)}{q^-(\xi_L) - q^+(0)} = \frac{T_L^4 + T_o^4}{2(T_L^4 - T_o^4)} + \varphi(\xi) \tag{45}$$

where $q^+(0) = \sigma T_o^4$ and $q^-(\xi_L) = \sigma T_L^4$, and

$$\varphi(\xi) = \frac{\left(\xi - \frac{\xi_L}{2}\right)\left[1 - (7 - 4\sqrt{3})e^{-\sqrt{12}\xi_L}\right] + \left(\frac{2 - \sqrt{3}}{2}\right)\left[e^{-\sqrt{12}(\xi_L - \xi)} - e^{-\sqrt{12}\xi}\right]}{\left(2 - \frac{1}{\sqrt{3}} - \xi_L\right) - \left(2 + \frac{1}{\sqrt{3}} + \xi_L\right)(7 - 4\sqrt{3})e^{-\sqrt{12}\xi_L}} \tag{46}$$

The normalized flux is

$$-\frac{q}{q^-(\xi_L) - q^+(0)} = \frac{\frac{4}{3}\pi h_o^+}{q^-(\xi_L) - q^+(0)} = \frac{4}{3}\left\{\frac{1}{1 + \xi_L + \left(1 - \frac{1}{\sqrt{3}}\right)\left[\frac{1 - (2 - \sqrt{3})e^{-\sqrt{12}\xi_L}}{1 - (7 - 4\sqrt{3})e^{-\sqrt{12}\xi_L}}\right]}\right\} \tag{47}$$

For the extreme values of ξ_L, the normalized flux is

$$\frac{-q}{q^-(\xi_L) - q^+(0)} = \begin{cases} 1\ , & \xi_L \ll 1 \\ \frac{4}{3\xi_L}\ , & \xi_L \gg 1 \end{cases} \tag{48}$$

Equation (46) has the advantage of being a relatively simple expression for the temperature distribution written in terms of an arbitrary optical thickness. It is, of course, possible to determine the expression similar to equation (46) for a higher approximation; but the complexity of the algebra increases rapidly with higher approximations.

Double P_5 Approximation

This particular approximation, which involves twelve differential equations and twelve unknowns, was chosen since it offered the possibility of a substantial increase in accuracy over the double P_1 approximation. The equations and boundary conditions are equations (11), (12), and (20) through (23) with $N = 5$. The method of solution follows that outlined in the previous sections. Unlike the previous example, the problem was not solved in terms of an arbitrary optical thickness. Instead, the problem was solved on an electronic computer for different specific values of ξ_L; therefore, only numerical results are available for the double P_5 approximation.

Discussion of Results

The normalized flux, $-q/[q^-(\xi_L) - q^+(0)]$, for the double P_1 and the double P_5 approximations is given in table I for different values of ξ_L. In reference 9 the problem of the plane absorbing layer was solved approximately by an integral equation approach; fluxes and temperature distributions were obtained by an iterative procedure. The values of normalized flux from reference 9 are also listed in table I for comparison. The maximum difference between any two comparable values in the table is less than 2 percent.

Values of the normalized flux are also plotted in figure 2, but because of the agreement among the values calculated by the different methods, only one curve is plotted, and it represents all the methods to within the accuracy of the graph.

The values from the double P_5 approximation calculation and those from the iterative calculation agree particularly well over most values of ξ_L except for a slight divergence at $\xi_L = 10$. Therefore, one is encouraged to assume that the values calculated are approximately correct since these two methods, one a differential equation method and one an integral equation method, are quite dissimilar.

Figure 3 contains curves of $\varphi(\xi)$ vs. ξ/ξ_L for the double P_1, double P_5, and iterative calculations for values of ξ_L from 0.1 to 10.0 The values for the double P_5 and the iterative calculations are in such close agreement that only one curve representing both is plotted in figure 3. The maximum difference between these two curves is approximately 1 percent. It is only necessary to plot $\varphi(\xi)$ for values from $\xi_L/2$ to ξ_L since $\varphi(\xi)$ is antisymmetric about $\xi_L/2$. The double P_1 curve (fig. 3) represents a cruder approximation and deviates slightly from the other curves.

The agreement among the three methods is not as good for small values of ξ_L. Figure 4 is a curve of $\varphi(\xi)$ vs. ξ/ξ_L for $\xi_L = 0.02$. In this case the double P_1 approximation is apparently failing to give a reasonably accurate result. The double P_5 result still agrees to within 5 percent with the iterative result.

The temperature distributions from the double P_5 approximation have also been compared with the curves published in reference 11. The values for these curves were obtained from a numerical solution of the integral equation using a desk calculator. The values from reference 11 and those from the double P_5 calculation agree to within the accuracy to which the curves in reference 11 can be read.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Aug. 6, 1964

APPENDIX A

SOLUTION OF DIFFERENTIAL EQUATIONS

LINEAR AND CONSTANT PART OF SOLUTION

The solution for $I_n^{\pm}$ was shown to be of the form of equation (17). To get the relations among the constants in the linear part of the solution the function $f_n^{\pm} + f_n^{\pm}\xi$ is substituted into equations (11) and (12). The following equations are obtained:

$$h_o^+ + h_1^+ + f_o^+ + h_o^+\xi - f_o^- - h_o^-\xi = 0 \tag{A1}$$

$$-h_o^- + h_1^- + f_o^- + h_o^-\xi - f_o^+ + h_o^+\xi = 0 \tag{A2}$$

and for $n \geq 1$

$$nh_{n-1}^+ + (2n + 1)h_n^+ + (n + 1)h_{n+1}^+ + 2(2n + 1)(f_n^+ + h_n^+\xi) = 0 \tag{A3}$$

$$nh_{n-1}^- - (2n + 1)h_n^- + (n + 1)h_{n+1}^- + 2(2n + 1)(f_n^- + h_n^-\xi) = 0 \tag{A4}$$

In equations (A1) through (A4) the constant parts and the parts involving ξ are equated separately to zero. From (A3) and (A4)

$$h_n^{\pm} = 0 \quad \text{for} \quad n \geq 1 \tag{A5}$$

$$f_n^{\pm} = 0 \quad \text{for} \quad n \geq 2 \tag{A6}$$

Then for $n = 1$, (A3) and (A4) are

$$h_o^+ + 6f_1^+ = 0 \tag{A7}$$

$$h_o^- + 6f_1^- = 0 \tag{A8}$$

From (A1) and (A2)

$$h_o^+ - h_o^- = 0 \tag{A9}$$

$$h_o^+ + f_o^+ - f_o^- = 0 \tag{A10}$$

There are six nonzero values of $h_n^{\pm}$ and $f_n^{\pm}$. Equations (A7) through (A10) are four relations among these constants. The other two relations needed to solve for the constants come from the boundary conditions.

EXPONENTIAL PART OF SOLUTION

To obtain the relations among the constants in the exponential part of the solution and to determine the exponents, λ_i, the function $a_i g_n^{\pm}(\lambda_i)e^{-\xi/\lambda_i}$ is substituted into equations (11) and (12). From equation (17) it can be seen that for this function $a_i g_n^{\pm}(\lambda_i)$ is equal to $d_n^{\pm}(\lambda_i)$. The coefficient $g_o^{+}(\lambda_i)$ is assigned the value unity for all i. The coefficients a_i, which are fixed by the boundary values for $I_n^{\pm}$, may be canceled from the equations which result from the above substitution. These equations are

$$-\frac{g_o^+}{\lambda_i} - \frac{g_1^+}{\lambda_i} + g_o^+ - g_o^- = 0 \qquad \text{for} \quad n = 0 \tag{A11}$$

$$\frac{g_o^-}{\lambda_i} - \frac{g_1^-}{\lambda_i} + g_o^- - g_o^+ = 0 \qquad \text{for} \quad n = 0 \tag{A12}$$

$$n g_{n-1}^{+} + (n + 1)g_{n+1}^{+} - (2\lambda_i - 1)(2n + 1)g_n^{+} = 0 \qquad \text{for} \quad 1 \leq n \leq N \tag{A13}$$

$$n g_{n-1}^{-} + (n + 1)g_{n+1}^{-} - (2\lambda_i + 1)(2n + 1)g_n^{-} = 0 \qquad \text{for} \quad 1 \leq n \leq N \tag{A14}$$

$$g_{n+1}^{+} = g_{n+1}^{-} = 0 \tag{A15}$$

It is helpful now to note that equations (A13) and (A14) have the same form as the recurrence formulas for two linearly independent sets of functions; the Legendre polynomials of the first kind, P_n, and the nonsingular part of the Legendre functions of the second kind, W_{n-1} (see ref. 3, p. 249). These relationships are

$$nP_{n-1}(2\lambda \mp 1) + (n + 1)P_{n+1}(2\lambda \mp 1) - (2\lambda \mp 1)(2n + 1)P_n(2\lambda \mp 1) = 0 \tag{A16}$$

$$nW_{n-2}(2\lambda \mp 1) + (n + 1)W_n(2\lambda \mp 1) - (2\lambda \mp 1)(2n + 1)W_{n-1}(2\lambda \mp 1) = 0 \tag{A17}$$

It is to be understood that the upper signs are to be used together and likewise the lower signs. Because of (A16) and (A17), it must be possible to write the $g_n^{\pm}$ as linear combinations of the P_n and the W_{n-1}.

$$g_n^{+}(\lambda_i) = A^{+}P_n(2\lambda_i - 1) - B^{+}W_{n-1}(2\lambda_i - 1) \tag{A18}$$

$$g_n^{-}(\lambda_i) = A^{-}P_n(2\lambda_i + 1) - B^{-}W_{n-1}(2\lambda_i + 1) \tag{A19}$$

Equations (A18) and (A19) define relations among the $g_n^{\pm}$ terms. If the constants A^+, A^-, B^+, and B^- and the λ_i are known, then it is possible to generate all of the values of $g_n^{\pm}$.

The assumption that g_o^+ is unity is equivalent to setting A^+ equal to unity. For $n \leq 0$, $W_{n-1}(2\lambda_i \pm 1)$ is set equal to zero. From equations (A18) and (A19) and the values of P_o, P_1, and W_o, the first few values of g_n are obtained:

$$\left.\begin{aligned} g_o^+ &= 1 \\ g_o^- &= A^- \\ g_1^+ &= 2\lambda_i - 1 - B^+ \\ g_1^- &= A^-(2\lambda_i + 1) - B^- \end{aligned}\right\} \tag{A20}$$

If equations (A20) are substituted into equations (A11) and (A12), the following two equations are obtained.

$$B^+ = B^- \tag{A21}$$

$$B^- = \lambda_i(A^- + 1) \tag{A22}$$

Two additional conditions on the constants B^+, A^-, and B^- are obtained from equation (A15).

$$P_{N+1}(2\lambda_i - 1) - B^-W_N(2\lambda_i - 1) = 0 \tag{A23}$$

$$A^-P_{N+1}(2\lambda_i + 1) - B^-W_N(2\lambda_i + 1) = 0 \tag{A24}$$

Equations (A21) through (A24) are sufficient to determine the three constants B^+, A^-, and B^- and the exponents λ_i.

Substitute equations (A21) and (A22) into equation (A24) and solve for A^-.

$$A^- = \frac{\lambda_i W_N(2\lambda_i + 1)}{P_{N+1}(2\lambda_i + 1) - \lambda_i W_N(2\lambda_i + 1)} \tag{A25}$$

From equation (A22)

$$B^- = \frac{\lambda_i P_{N+1}(2\lambda_i + 1)}{P_{N+1}(2\lambda_i + 1) - \lambda_i W_N(2\lambda_i + 1)} = B^+ \tag{A26}$$

Equations (A23) and (A26) combine to give

$$P_{N+1}(2\lambda_i - 1)P_{N+1}(2\lambda_i + 1) - \lambda_i\Big[P_{N+1}(2\lambda_i - 1)W_N(2\lambda_i + 1)$$

$$+ P_{N+1}(2\lambda_i + 1)W_N(2\lambda_i - 1)\Big] = 0 \qquad \text{(A27)}$$

Equation (A27) is a polynomial which defines the 2N exponents λ_i in the solution (eq. (17)). As proved previously all λ_i are real. Although equation (A27) apparently has $2(N + 1)$ roots, it is found that the coefficients of the two highest power terms in the polynomial always cancel. This can be easily seen by substituting the general terms for P_n and W_{n-1} (see, e.g., ref. 12) into (A27). Note that there are separate values of B^+, B^-, and A^- associated with each λ_i.

In summary, it is found that equation (A27) defines the exponents λ_i. A value of B^+, A^-, and B^-, and a complete set of $g_n^{\pm}(\lambda_i)$ correspond to each value of λ_i. The expression for $I_n^{\pm}$ is

$$I_n^{\pm}(\xi) = f_n^{\pm} + h_n^{\pm}\xi + \sum_{i=1}^{2N} a_i g_n^{\pm}(\lambda_i)e^{-\xi/\lambda_i} \qquad \text{(A28)}$$

Of the six nonzero constants, $f_n^{\pm}$ and $h_n^{\pm}$, four may be considered as determined in terms of the other two, due to equations (A7) through (A10). The two undetermined constants, $f_n^{\pm}$ and $h_n^{\pm}$, and the 2N values of a_i are determined by the boundary conditions. Equations (20) through (23) are the necessary $2(N + 1)$ boundary conditions.

REFERENCES

1. Kourganoff, V.: Basic Methods in Transfer Problems. Clarendon Press, Oxford, 1952.

2. Yvon, J.: La Diffusion Macroscopique Des Neutrons Une Methode D'Approximation. Journal of Nuclear Energy, vol. 4, 1957, pp. 305-318.

3. Weinberg, Alvin M., and Wigner, Eugene P.: The Physical Theory of Neutron Chain Reactors. University of Chicago Press, Chicago, 1958.

4. Shiff, D., and Ziering, S.: Yvon's Method for Slabs. Nuclear Science and Engineering, vol. 3, 1958, pp. 635-647.

5. Bengston, J.: Neutron Self-Shielding of a Plane Absorbing Foil. Oak Ridge National Lab. MEMO CF-56-3-170, 1956.

6. Shiff, D., and Ziering, S.: Many Fold Moment Method. Nuclear Science and Engineering, vol. 7, 1960, pp. 172-183.

7. Chandrasekhar, S.: Radiative Transfer. Dover Pub., Inc. N. Y., 1960.

8. Ferrar, W. L.: Algebra. Second ed., Oxford University Press, London, 1957.

9. Heaslet, Max. A., and Fuller, Franklyn B.: Approximate Predictions of the Transport of Thermal Radiation Through an Absorbing Layer. NASA TN D- , 1964.

10. Goulard, R., and Goulard, M.: One-Dimensional Energy Transfer in Radiant Media. Int. Jour. of Heat and Mass Transfer, vol. 1, 1960, pp. 81-91.

11. Usiskin, C. M., and Sparrow, E. M.: Thermal Radiation Between Parallel Plates Separated by an Absorbing-Emitting Nonisothermal Gas. Int. Jour. of Heat and Mass Transfer, vol. 1, 1960, pp. 28-36.

12. Jahnke, Eugene, Emde, Fritz, and Losch, Friedrich: Tables of Higher Functions. Sixth ed., McGraw-Hill Book Co., N. Y., 1960.

TABLE I.- NORMALIZED FLUX $\left\{-q/[q^-(\xi_L) - q^+(0)]\right\}$

L	Double P_1	Double P_5	Iterative (ref. 9)
0.02	0.980579	0.980864	0.980963
.1	.912691	.915692	.915710
.5	.698888	.704152	.704093
1.0	.550953	.553404	.553867
2.0	.389571	.390058	.390663
3.0	.301479	.301644	.301770
5.0	.207599	.207657	.206538
10.0	.116727	.116745	.114784

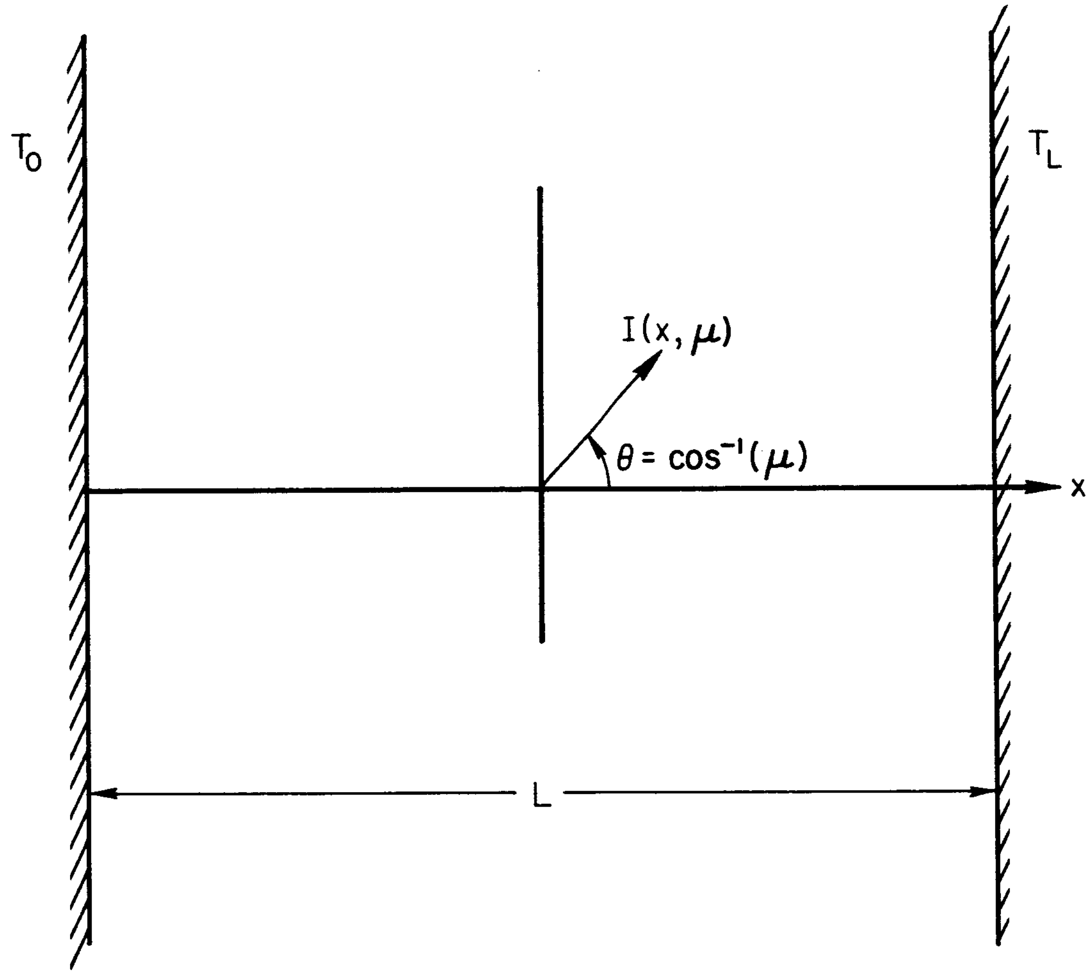

Figure 1.- Parallel walls separated by absorbing medium.

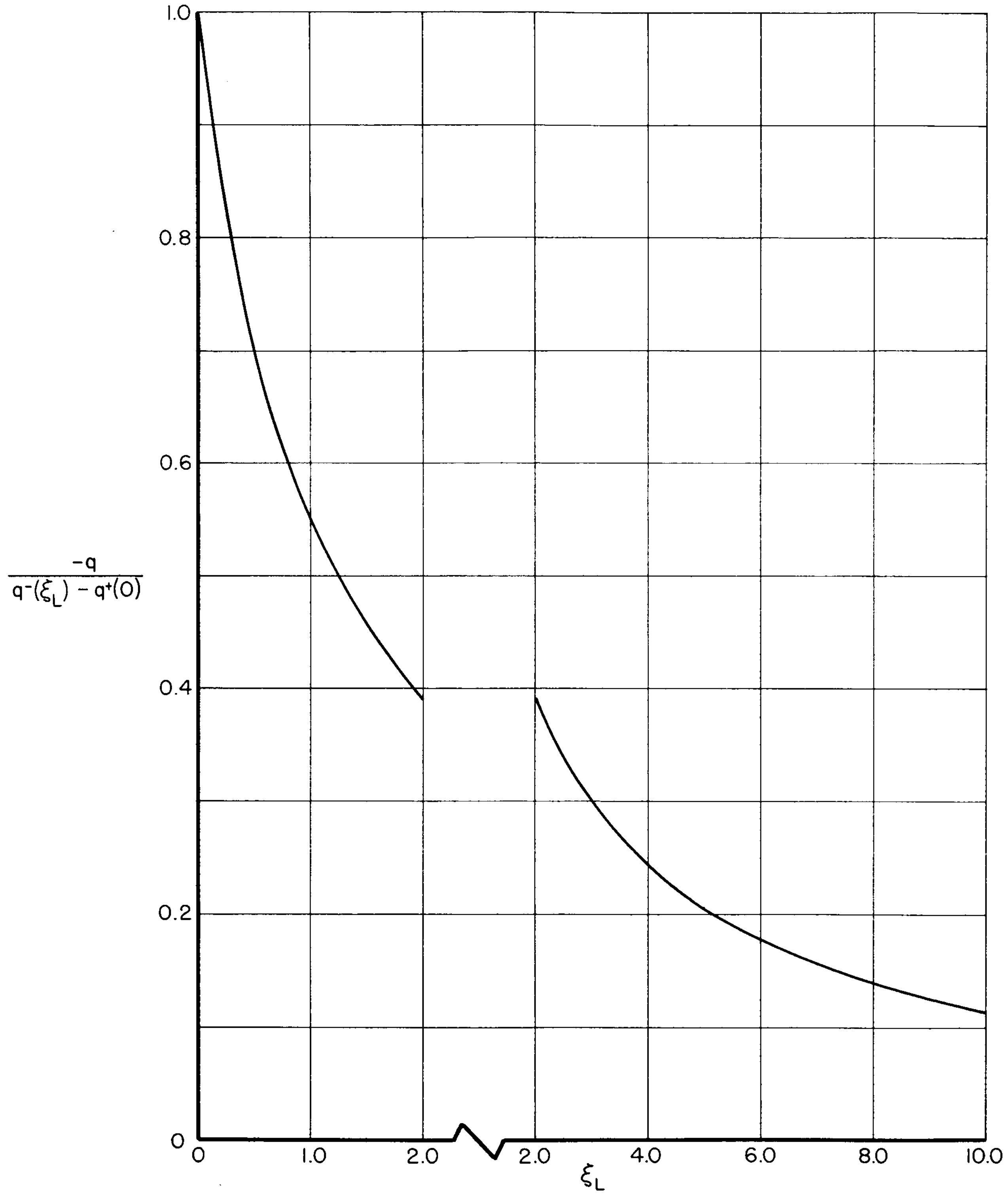

Figure 2.- Normalized flux vs. optical thickness.

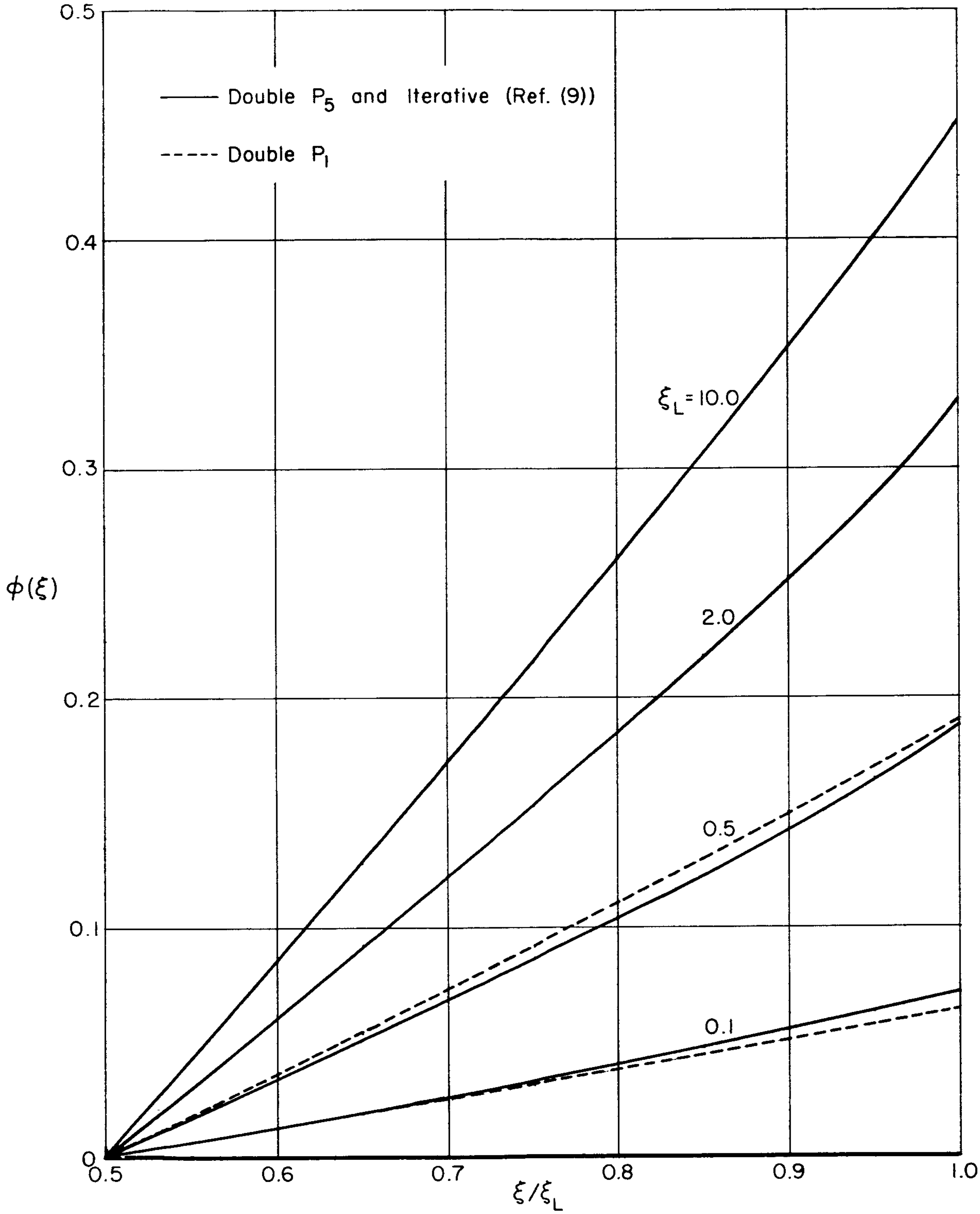

Figure 3.- Dimensionless emission function, $\varphi(\xi)$, vs. normalized optical depth, ξ/ξ_L.

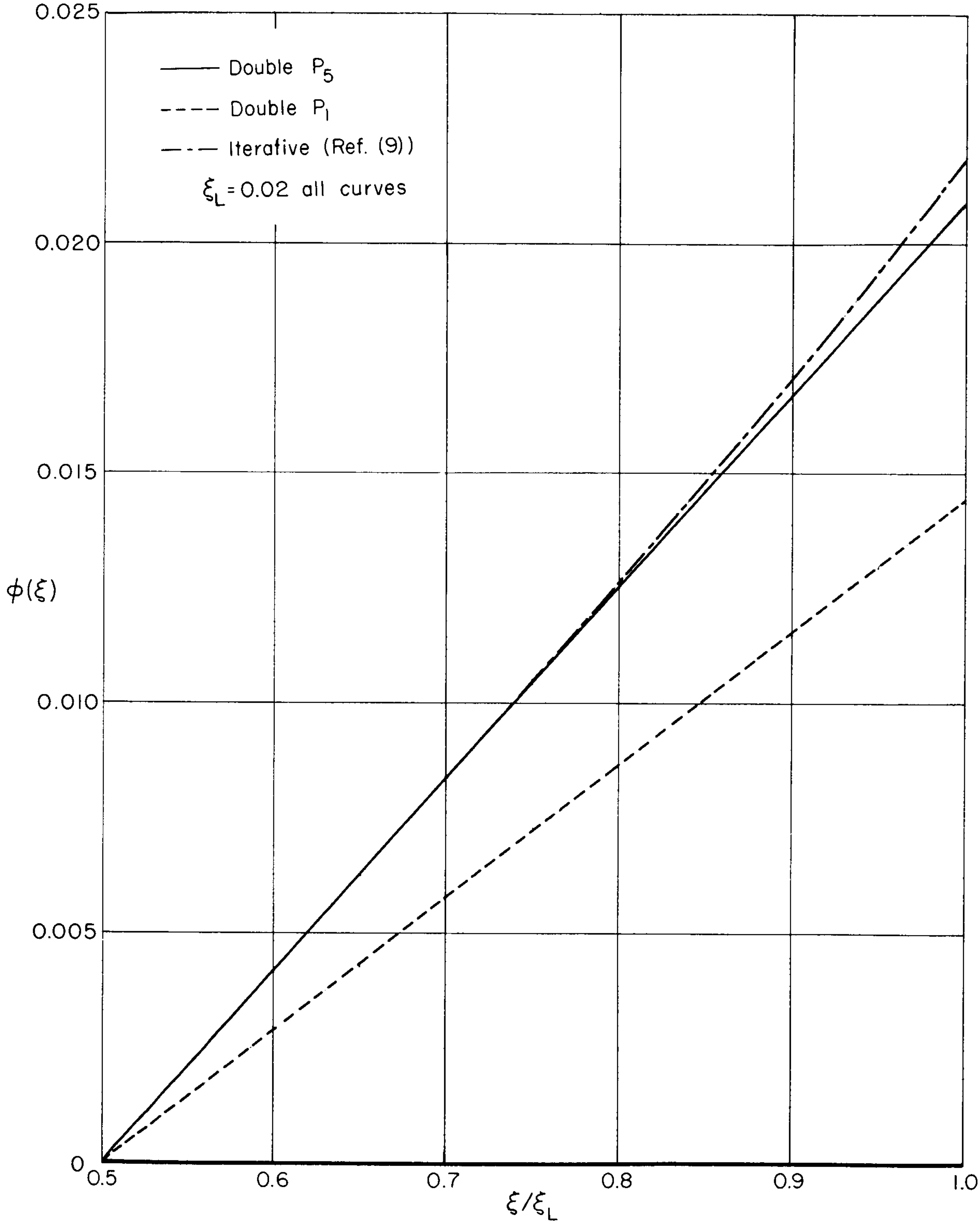

Figure 4.- Dimensionless emission function, $\varphi(\xi)$, vs. normalized optical depth, ξ/ξ_L.

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